

Calculus III: Coursework 1
M.A.H. MacCallum 2007

Because this sheet was issued later than intended, the deadline will be 5PM on Wednesday Oct 3rd. Normal deadlines will be 5PM on Tuesdays. Hand-in is to the green box in the basement of the Maths building.

1. Give the common name for the curve

$$x = \phi \cos \phi \quad y = \phi \sin \phi$$

with a brief explanation or sketch. [The common name should be obvious from a sketch.] [10]

Answer

The curve is a spiral, starting at the origin and going anti-clockwise as it moves outward. (A bit like a circle, but with increasing radius as you increase ϕ .) Each time the curve goes round the origin it moves out by the same amount, 2π .

Marks: Full marks for any sensible and reasonably complete description.

2. Describe each of the following surfaces in three dimensional space. [For example, a suitable description of $x^2 + y^2 = 1$ would be that it is a cylinder of radius 1 whose axis is the z axis.] [18]

(a) $z = 0$

(b) $x^2 + y^2 + z^2 - 2y - 4z + 1 = 0$

(c) $x^2 + y^2 = z$

Answer

The surfaces are as follows.

(a) The x, y plane.

(b) A sphere of radius 2, centred at $(0, 1, 2)$.

(c) A circular paraboloid (bowl-shaped) surface, (sketch would show it symmetric about the z -axis, above, and touching, the x, y plane).

Marks: (a) 4 (b) 7 (c) 7.

3. Simplify the following [12]

(a) $\cos 120^\circ$

(b) $\sin 75^\circ \sin 15^\circ$

Answer

We have

- (a) $\cos 120^\circ = -\sin 30^\circ = -\frac{1}{2}$
 (b) $\sin 75^\circ \sin 15^\circ = \frac{1}{2}(\cos 60^\circ - \cos 90^\circ) = \frac{1}{4}$

Marks: Note there are many correct routes here. 4+4 for using correct formulas (or 2+2 for substituting half-correct formulas); 2+2 for inserting correct values.

4. Find $\int \sin^4 x \, dx$ and $\int_0^{2\pi} \sin^4 x \, dx$. [15]

Answer

We have

$$\begin{aligned} \sin^4 x &= \frac{1}{4}(1 - \cos 2x)^2 \\ &= \frac{1}{4} - \frac{1}{2} \cos 2x + \frac{1}{4} \cos^2 2x \\ &= \frac{1}{4} - \frac{1}{2} \cos 2x + \frac{1}{8}(1 + \cos 4x) \\ &= \frac{3}{8} - \frac{1}{2} \cos 2x + \frac{1}{8} \cos 4x \end{aligned}$$

which gives

$$\frac{3}{8}x - \frac{1}{4} \sin 2x + \frac{1}{32} \sin 4x [+constant]$$

for the indefinite integral and $\frac{3}{4}\pi$ for the definite integral.

Marks: 6 for expanding and doing so correctly, 4 for integration, 3 for realizing that the periodic terms fall out of the definite integral, 2 for its final answer. Some students may recall and use the recurrence formula for integrals of $\sin^n x \cos^m x$: they should get

$$\frac{3}{8}x - \frac{3}{8} \sin x \cos x - \frac{1}{4} \sin^2 x \cos^2 x [+constant]$$

for the indefinite integral. No penalty for omitting the constant this time.

5. Show that for positive integers m, n [15]

$$\begin{aligned} \int_0^{2\pi} \sin mx \sin nx \, dx &= \pi \delta_{mn} \\ \int_0^{2\pi} \cos mx \cos nx \, dx &= \pi \delta_{mn} \end{aligned}$$

where

$$\delta_{mn} \equiv \begin{cases} 1 & \text{if } m = n \\ 0 & \text{otherwise.} \end{cases}$$

Answer

We have

$$\begin{aligned} \int_0^{2\pi} \sin mx \sin nx \, dx &= \frac{1}{2} \int_0^{2\pi} (\cos[(m-n)x] - \cos[(m+n)x]) \, dx \\ &= \begin{cases} \left[\frac{\sin[(m-n)x]}{m-n} - \frac{\sin[(m+n)x]}{m+n} \right]_0^{2\pi} & \text{if } m \neq n \\ \left[x - \frac{\sin(2mx)}{2m} \right]_0^{2\pi} & \text{if } m = n \end{cases} \end{aligned}$$

$$\begin{aligned}\int_0^{2\pi} \cos mx \cos nx \, dx &= \frac{1}{2} \int_0^{2\pi} (\cos[(m-n)x] + \cos[(m+n)x]) \, dx \\ &= \begin{cases} \left[\frac{\sin[(m-n)x]}{m-n} + \frac{\sin[(m+n)x]}{m+n} \right]_0^{2\pi} & \text{if } m \neq n \\ \left[x + \frac{\sin(2mx)}{2m} \right]_0^{2\pi} & \text{if } m = n \end{cases}.\end{aligned}$$

The only surviving term in either will be $\frac{1}{2} \int_0^{2\pi} \cos[(m-n)x] \, dx = \frac{1}{2}[x]_0^{2\pi}$ if $m = n$, giving π .

Marks: 5 for basic manipulation of integrand, 4 for general case integration, 4 for dealing with $m = n$ correctly, 2 for correct completion.

6. Show that [10]

$$\ln(\cosh x + \sinh x) + \ln(\cosh x - \sinh x) = 0$$

Answer

There are many correct ways, e.g. using $\cosh x + \sinh x = e^x$, $\cosh x - \sinh x = e^{-x}$ to get $x - x = 0$ or

$$\begin{aligned}\ln(\cosh x + \sinh x) + \ln(\cosh x - \sinh x) &= \ln[(\cosh x + \sinh x)(\cosh x - \sinh x)] \\ &= \ln(\cosh^2 x - \sinh^2 x) \\ &= \ln 1 = 0.\end{aligned}$$

Marks: 5 for any correct method, 5 for its correct application.

7. Work out the angle between the vectors $2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $4\mathbf{i} + \mathbf{j} - \mathbf{k}$.

Calculate the vector product (cross product) of these two vectors. [20]

Answer

The dot product is 9 while the magnitudes are 3 and $3\sqrt{2}$, which gives $\cos \theta = 1/\sqrt{2}$ so the angle is $\pi/4$ or 45° .

The cross-product is given by

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 1 \\ 4 & 1 & -1 \end{vmatrix} = -3\mathbf{i} + 6\mathbf{j} - 6\mathbf{k}$$

Marks: Knowing dot-product (or cross product which could also give the angle) is involved and using it 4, correct calculations 6. 4 for method for cross-product, and 2 per correct component.